A CONTACT PROBLEM IN THE THEORY OF ELASTICITY WITH A SINGLE CONTROLLING PARAMETER

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We consider a contact problem in the theory of elasticity in a general formulation. We give a nonlinear equation for determining the boundary of the contact region and some formulas connecting the fundamental physical quantities with a characteristic parameter of the contact region.

The contact problem is constantly attracting the attention of theoreticians and engineers. Along with purely mechanical applications, the study of processes of heat exchange and electrical conductivity at the contact of solid deformed bodies requires that we know the contact region and the distribution of contact pressure. The latter is necessary, for example, in determining the true contact area, which consists of the contact regions of the microuniformities, and consists of the total of only a small part of the macroscopic region. Unlike the mixed boundary-value problem, the contact problem in elasticity theory has an unknown dividing line for the boundary conditions, which also represents the principal difficulty in solving the three-dimensional problem. This problem was investigated in the general formulation in [1], where the problem is divided into a series of linear problems on plane stamps and a nonlinear system for determining their contours. In the present article instead of this we present a nonlinear equation and also some formulas that give, in quadratures, the dependence of the magnitude of the penetration of the stamp on a characteristic parameter of the contact region. Although we consider only the contact of a stamp with an elastic body below, all the arguments can also be extended to the case of the contact of two elastic bodies.

1. We consider the contact of a smooth rigid stamp with an elastic body. Assuming the displacement of each point $M(\mathbf{r})$ of the surface of the body to be on a tangent to the surface, and assuming parallel displacement of the stamp, we denote the latter by $w(\mathbf{r}, \lambda)$. Here λ is a parameter that determines the contact process. It can be, e.g., a general contact force, or the size of the contact region. There can also be several determining physical parameters, when, for example, several unconnected stamps penetrate the body, but in the present article we will not consider this case. Just as was done in [1], we take $\lambda = \alpha$, where α is the penetration of the stamp.

The boundary conditions of the formulated problem can be written in the form

$$\begin{cases} w(\mathbf{r}, \alpha) = \alpha - f(\mathbf{r}), & \mathbf{r} \in S_\alpha, \\ p(\mathbf{r}, \alpha) = 0, & \mathbf{r} \,\bar{\in}\, S_\alpha, \end{cases} \tag{1}$$

$$\begin{cases} w(\mathbf{r}, \alpha) > \alpha - f(\mathbf{r}), & \mathbf{r} \,\bar{\in}\, S_\alpha, \\ p(\mathbf{r}, \alpha) \geqslant 0, & \mathbf{r} \in S_\alpha. \end{cases} \tag{2}$$

Here p is the pressure component normal to the surface of the body (the tangential component equals zero), S_α is the contact region, $f(\mathbf{r})$ is the initial distance between the surfaces of the body and the stamp in the direction of displacement of the stamp.

Let there be a known displacement $K(\mathbf{r}, \mathbf{r}')$ of the point $M(\mathbf{r})$ in the direction of motion of the stamp, generated under the action of a normal concentrated force, applied at the point $M(\mathbf{r}')$. Then the displacement $w(\mathbf{r}, \alpha)$, caused by the distributed contact pressure $p(\mathbf{r}, \alpha)$, is obtained by the integration

$$w(\mathbf{r}, \alpha) = \int\limits_{S_\alpha} K(\mathbf{r}, \mathbf{r}')\cdot p(\mathbf{r}', \alpha)\, d\sigma. \tag{3}$$

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The boundary-value problem with boundary conditions (1) and (2) therefore reduces to the system

$$\begin{cases} \int\limits_{S_\alpha} K(\mathbf{r}, \mathbf{r}') p(\mathbf{r}', \alpha) d\sigma = \alpha - f(\mathbf{r}), & \mathbf{r} \in S_\alpha, \\ p(\mathbf{r}, \alpha) = 0, & \mathbf{r} \bar{\in} S_\alpha, \end{cases} \tag{4}$$

$$\begin{cases} \int\limits_{S_\alpha} K(\mathbf{r}, \mathbf{r}') p(\mathbf{r}', \alpha) d\sigma > \alpha - f(\mathbf{r}), & \mathbf{r} \bar{\in} S_\alpha, \\ p(\mathbf{r}, \alpha) \geqslant 0, & \mathbf{r} \in S_\alpha, \end{cases} \tag{5}$$

which, for each value of α, determines the contact pressure and the contact region S_α.

As is shown in [1], the solution of the problem (4)-(5) is represented by the integrals

$$p(\mathbf{r}, \alpha) = \int\limits_0^\alpha p_0(\mathbf{r}; [\Gamma_\lambda]) d\lambda = \int\limits_{\alpha_r}^\alpha p_0(\mathbf{r}; [\Gamma_\lambda]) d\lambda, \tag{6}$$

$$w(\mathbf{r}, \alpha) = \int\limits_0^\alpha w_0(\mathbf{r}; [\Gamma_\lambda]) d\lambda, \tag{7}$$

where Γ_α is the contour of the region S_α, α_r is the value of α at which the point $M(\mathbf{r})$ falls within the contact region, p_0 and w_0 are the solution of the problem of an adjoining particular stamp with base S_α. By "adjoining" we have in mind a stamp whose surface coincides with the initial surface of the body. The solution of this problem is proportional to the value of its penetration. The quantities p_0 and w_0 correspond to a penetration to a particular depth; therefore, for brevity, below, we call such stamps simply "particular" stamps. In the simplest case of an elastic half-space, e.g., particular stamps, we have simply plane stamps, and the representation (6), (7) describes a rather clear model. We must emphasize that p_0 and w_0 do not depend directly on α, and are functions of the line Γ_α, which is denoted by square brackets. For each function $f(\mathbf{r})$ given beforehand, the contact process is described by a series of particular stamps that, for each α, satisfy the system

$$\begin{cases} \int\limits_{S_\alpha} K(\mathbf{r}, \mathbf{r}') p_0(\mathbf{r}'; [\Gamma_\alpha]) d\sigma = 1, & \mathbf{r} \in S_\alpha, \\ p_0(\mathbf{r}; [\Gamma_\alpha]) = 0, & \mathbf{r} \bar{\in} S_\alpha. \end{cases} \tag{8}$$

It is easy to see that by using the representation (6)-(7), we at once satisfy the second equation in (4), and the conditions for the displacement can be written in the form

$$\int\limits_0^\alpha w_0(\mathbf{r}; [\Gamma_\lambda]) d\lambda = \alpha - f(\mathbf{r}), \quad \mathbf{r} \in S_\alpha, \tag{9}$$

$$\int\limits_0^\alpha w_0(\mathbf{r}; [\Gamma_\lambda]) d\lambda > \alpha - f(\mathbf{r}), \quad \mathbf{r} \bar{\in} S_\alpha. \tag{10}$$

Thus, the nonlinear contact problem is divided into two parts: determination of the boundary of the contact region in the loading process from the nonlinear system (9)-(10), and a series of problems on particular stamps (8) with contours that are already given. The solutions of the latter can then be integrated with respect to the penetration parameter. In [1] we present a calculation method for determining the boundaries from the system (9)-(10). Essentially it is a numerical solution of the equation

$$\int\limits_0^\mu w_0(\mathbf{R}; [\Gamma_\lambda]) d\lambda = \mu - f(\mathbf{R}), \quad \mathbf{R} \in \Gamma_\mu. \tag{11}$$

We assume that Γ_μ satisfies Eq. (11) for all $0 < \mu \leq \alpha$. Then it is not difficult to verify that condition (9) is satisfied. Investigating the asymptotic form of the specific solution near the boundary, we can show that the solutions Γ_μ $(0 < \mu \leq \alpha)$, which, being deformed during the loading process, fill the entire real contact region, and satisfy (10), at least in the neighborhood of the boundary Γ_α. Thus, system (9)-(10) can be replaced by Eq. (11) for the entire family Γ_μ.

We note that equations of form (6), (7) in the axisymmetric case for a half-space were given in [2], and to obtain the relation between the penetration and the radius of the contact circle, an equation of the type (11) was used there.

2. Using the representation (6)-(7), we obtain formulas that are important for applications.

The total force of contact of a particular stamp, having the contour Γ_α, equals

$$Q(\alpha) = \int\limits_{S_\alpha} p_0(\mathbf{r};\ [\Gamma_\alpha]) \cos(\mathbf{n},\ \bar{\alpha})\, d\sigma,$$

where **n** is the vector of the normal to the surface of an elastic body, $\bar{\alpha}$ is the displacement vector of the stamp. The total force of an arbitrary stamp equals

$$P(\alpha) = \int\limits_{S_\alpha} p(\mathbf{r},\ \alpha) \cos(\mathbf{n},\ \bar{\alpha})\, d\sigma.$$

From (6) we have

$$P(\alpha) = \int\limits_0^\alpha Q(\lambda)\, d\lambda. \tag{12}$$

Let the elastic body be fastened along part of the surface. The elastic energy of the stressed state that arises during contact equals

$$\frac{1}{2}\int\limits_{S_\alpha} p(\mathbf{r},\ \alpha)\, w(\mathbf{r},\ \alpha) \cos(\mathbf{n},\ \bar{\alpha})\, d\sigma = \frac{1}{2}\alpha \int\limits_{S_\alpha} p(\mathbf{r},\ \alpha) \cos(\mathbf{n},\ \bar{\alpha})\, d\sigma$$
$$-\frac{1}{2}\int\limits_{S_\alpha} p(\mathbf{r},\ \alpha) f(\mathbf{r}) \cos(\mathbf{n},\ \bar{\alpha})\, d\sigma = \frac{1}{2}\left\{\alpha P(\alpha) - \int\limits_0^\alpha N(\lambda)\, d\lambda\right\},$$

where

$$N(\alpha) = \int\limits_{S_\alpha} p_0(\mathbf{r};\ [\Gamma_\alpha]) f(\mathbf{r}) \cos(\mathbf{n},\ \bar{\alpha})\, d\sigma.$$

During penetration of the stamp, the work

$$\int\limits_0^\alpha P(\lambda)\, d\lambda$$

is done, equal to the elastic energy

$$\alpha P(\alpha) - \int\limits_0^\alpha N(\lambda)\, d\lambda = 2\int\limits_0^\alpha P(\lambda)\, d\lambda.$$

Differentiating the last relation with respect to α and using (12), we obtain

$$P(\alpha) = \alpha Q(\alpha) - N(\alpha). \tag{13}$$

If we know the family of contours Γ_α depending on some parameter a and satisfying Eq. (11), then we can obtain simply the dependence of α on a. The parameter a for the case of a circle can be the radius; for the case of an ellipse it can be the semiaxis, etc. We have

$$P(a) = \int\limits_0^a Q(t)\, \frac{d\alpha(t)}{dt}\, dt = \alpha(a)\, Q(a) - N(a).$$

Differentiating this equation with respect to a, taking into account that α, P, Q, and N are now already functions of a, we obtain

$$-N'(a) + \alpha(a)\, Q'(a) = 0,$$
$$\alpha(a) = \frac{N'(a)}{Q'(a)}. \tag{14}$$

This important formula is put in correspondence with the contour Γ_a by the penetration α. Having the solution p_0 and w_0 for a definite family of contours Γ_α, from Eqs. (6), (7), (13), and (14) we can obtain the solution for an arbitrary function f in quadratures. Although the displacements w_0 do not appear in the expressions for α, p, and P, we must know them in order to check the fact that the selected family of contours Γ_a satisfies Eq. (11), i.e., it is realized in the contact process. If we consider the one-dimensional case, and the region of contact is completely described by a single parameter, then the solution is immediately expressed in terms of p_0 from Eqs. (6), (13), and (14), and Eq. (11) does not require checking. In the axisymmetric case, for the parameter a we can take the radius of the contact circle. For example, for an elastic half-space and stamp of arbitrary form f(r) it is easy to write out in quadratures the fundamental formulas, obtained earlier by more complex methods (Shtaerman [3], Galin [4], et al.). The quantities p_0 and w_0 have the form

$$p_0(r,\ a)=\frac{2A}{\pi\sqrt{a^2-r^2}},\quad r\leqslant a,\quad w(r,\ a)=\frac{2}{\pi}\arcsin\frac{a}{r},\quad r\geqslant a, \tag{14}$$

where $A = G/(1-\nu)$, G is the shear modulus, and ν is the Poisson ratio. We obtain:

$$N(a)=\int_0^{2\pi}d\theta\int_0^a p_0(r,\ a)f(r)\,dr=4Aa\int_0^1\frac{f(ax)\,x\,dx}{\sqrt{1-x^2}},\quad Q(a)=4Aa,$$

$$\frac{dN(a)}{da}=4A\int_0^1\frac{[f(ax)+axf'(ax)]\,x\,dx}{\sqrt{1-x^2}},\qquad \frac{dQ(a)}{da}=4A,$$

$$\alpha(a)=\int_0^1\frac{[f(ax)+axf'(ax)]\,x\,dx}{\sqrt{1-x^2}},$$

$$\frac{d\alpha(a)}{da}=\int_0^1\frac{[2f'(ax)+axf''(ax)]\,x^2dx}{\sqrt{1-x^2}},$$

$$p(r,\ a)=\int_r^a p_0(r,\ \xi)\frac{d\alpha(\xi)}{d\xi}\,d\xi=\frac{2A}{\pi}\int_r^a d\xi\int_0^1 dx\,\frac{[2f'(\xi x)+\xi xf''(\xi x)]\,x^2}{\sqrt{(\xi^2-r^2)(1-x^2)}},$$

$$r\leqslant a,$$

$$w(r,\ a)=\int_0^a w_0(r,\ \xi)\frac{d\alpha(\xi)}{d\xi}\,d\xi=\frac{2}{\pi}\int_0^a d\xi\int_0^1 dx\arcsin\frac{\xi}{r}$$

$$\times\ \frac{[2f'(\xi x)+\xi xf''(\xi x)]\,x^2}{\sqrt{1-x^2}},\quad r\geqslant a.$$

When the elastic body has a plane surface, by substituting (14) into (6), we obtain a particular case of the formula of M. G. Krein [5] for solution of one-dimensional integral equations. We should note that the expression for the total force (13) was obtained earlier by a different method for elastic half-spaces [6] and a plane layer [7].

NOTATION

R, **r**, and **r'**	are the radius vectors;
w, p, w_0, and p_0	are the displacements of the points of the surface of the elastic body and the pressure under the stamp for a stamp of arbitrary and particular shape, respectively;
f	is a function determining the shape of the stamp;
P and Q	are the total forces of contact for a stamp of arbitrary and particular shape;
N	is the component of the total force;
α, μ, and λ	denote the value of the penetration of the stamp;
S_α	is the contact region;
Γ_α	is its contour;
K	is the kernel of the integral equation;
n	is the vector of the normal to the surface of the elastic body;
a	is a characteristic parameter of the contact region, in particular, the radius of a circle;
t, x, and ξ	are variables of integration;
$d\sigma$	is an element of area;
$A = G/(1-\nu)$;	
G	is the shear modulus;
ν	is the Poisson ratio.

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